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Group-theoretic aspects of the Hosotani mechanism

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Abstract. If a gauge configuration has a non-trivial holonomy group Φ in the vacuum, then the gauge symmetry is broken to the centraliser of Φ in the gauge group. This approach to symmetry breaking has found important applications recently. Here we study this mechanism from a general point of view: given a compact Lie group G and a compact subgroup H, we find conditions for the existence of a group J with $\mathrm{H} = C\mathrm{J}$ (centraliser of J), study the uniqueness of J, and ask whether J can be represented as the holonomy group of some connection.

1. General remarks on the Hosotani mechanism

Superstring theory [1] gives rise, in a natural way, to very large gauge groups such as E_8. It is fortunate that these symmetries need not be broken by means of the Higgs mechanism. In the case of E_8, for example, the vacuum gauge configuration is such that the holonomy group is SU(3) (at least in the simplest cases); if this SU(3) is embedded in E_8 in the most natural way, then the latter is broken down to E_6, which becomes the grand unification symmetry. The corresponding connection is flat, but can have a non-trivial holonomy group which again serves to break the E_6 symmetry down to the strong plus electroweak group.

This 'Hosotani [2] mechanism' is of general interest, independent of these particular applications [3], and it is from this point of view that we wish to study it here. For the sake of organisation, we have arranged the results around one main question: given a compact Lie group G and a compact subgroup H, how can the Hosotani mechanism be used to break G down to H? First, we must find another subgroup $\mathrm{J} \subseteq \mathrm{G}$ such that $C\mathrm{J} = \mathrm{H}$, where $C\mathrm{J}$, the centraliser of J in G, is defined by $C\mathrm{J} = \{g \in \mathrm{G}$ such that $gj = jg$ for every $j \in \mathrm{J}\}$. This is necessary because the group of vertical automorphisms of a principal bundle preserving a given connection is isomorphic to the centraliser of the latter's holonomy group. Secondly, we must show that there exists a principal bundle over the base manifold M, with a connection having J as the holonomy group. If, as sometimes happens, no such bundle exists, then we must ask whether there is some other group $\bar{\mathrm{J}}$ with $C\bar{\mathrm{J}} = \mathrm{H}$; that is, we must consider uniqueness as well as existence.

One soon finds that, quite often, J does not exist; in other cases, there exists no principal bundle (over a specific base manifold) with a connection having J as holonomy group, for all J with $C\mathrm{J} = \mathrm{H}$. Under these circumstances, G *cannot* be broken to H by means of the Hosotani mechanism. For example, let $\mathrm{G} = \mathrm{SU}(n)$, $\mathrm{H} = \mathrm{SU}(m)$, $m < n$, for any embedding. Then we shall show that $\mathrm{SU}(n)$ has no subgroup of which $\mathrm{SU}(m)$ is the centraliser; that is, $\mathrm{SU}(n)$ cannot be broken to $\mathrm{SU}(m)$ by the Hosotani mechanism. There are many such examples. This is to be constrasted with the case of the Higgs mechanism. According to the Palais-Mostow theorem [4, 5], every compact subgroup of a compact Lie group G occurs as the isotropy group of some point in some representation space of G: thus, in principle, every such subgroup can be obtained via the Higgs mechanism. (In reality, of course, there are limitations of a physical nature—for example, the Higgs Lagrangian must be such as to lead to a renormalisable theory. It is important, however, to distinguish these restrictions from basic questions of mathematical principle.)

This lack of flexibility certainly limits the applicability of the Hosotani mechanism. For example, we shall find that the Weinberg-Salam electroweak symmetry cannot be broken to the electromagnetic U(1) in this way. On the other hand, it can be argued that a certain degree of inflexibility is a virtue; certainly, conventional gauge theory presents an unwelcome variety of permissible choices of gauge groups and symmetry breaking patterns. Furthermore, the Hosotani mechanism only makes use of material already (implicitly) present in gauge theory—it does not introduce extraneous items such as scalar fields. Apart from its intrinsic attractions, this property may point the way to a more complete understanding of the origin of symmetry breaking in general. For example, in superstring theory, the process of 'embedding the spin connection in the gauge connection' (which leads, via the Hosotani mechanism, to the breaking of E_8 to E_6) arises from anomaly cancellation conditions, yielding a deep explanation of symmetry breaking, at least in this instance.

A more serious drawback of this method is the following fact. Given G and a subgroup J, one *cannot* compute CJ until the embedding of J in G is specified. For example, the centraliser of SU(3) in E_8 is, indeed, E_6—*provided* that SU(3) is embedded in E_8 in a particular way. But with a different embedding, the centraliser can be a certain disconnected group locally isomorphic to SU(2). Obviously this would be totally unacceptable for the superstring application. Now it can certainly be argued that, precisely because the centraliser is much 'larger' in the former case than in the latter, the first embedding is more natural than the second; and, as we shall see, this idea can (in a rough way) be quantified. Nevertheless this fact does detract somewhat from the 'naturalness' of this approach.

Here we shall not attempt to explain why some embeddings are (physically) preferred to others. We raise this point mainly for technical reasons: clearly, no general investigation of the Hosotani mechanism can proceed very far unless some conditions are imposed on the type of embedding under consideration. A very simple quantitative scheme for distinguishing 'good' embeddings from 'bad' will be discussed below.

Our results are divided into three sections: first, on the existence and uniqueness of 'solutions' of the equation $C\mathrm{J} = \mathrm{H}$; second, on the existence of a principal bundle (over a given manifold M) having a connection with J as holonomy group; and third, on specific examples.

1.1. Notation

If A is a subgroup of B, we denote the centraliser of A in B by $C_{\mathrm{B}}\mathrm{A}$. The subscript is dropped if B = G, i.e. $C \equiv C_{\mathrm{G}}$. The *centre* of any group H, denoted ZH, is defined by $Z\mathrm{H} = C_{\mathrm{H}}\mathrm{H}$. Let G be a disconnected Lie group. The component containing the identity is a subgroup of G, usually denoted G_0. Unfortunately, this notation is not convenient here; we shall use the symbol EG instead. Notations such as ECZH mean, of course, E(C(Z(H))). The word 'subgroup' should always be interpreted as 'closed subgroup' in the Lie group context.

2. Breaking a gauge group to a given subgroup

The fundamental result on which the mechanism is based is the following theorem.

Theorem 1. Let (P, M, G) be a principal bundle over a base manifold M, with structural group G, and with a connection having holonomy group Φ. Then the group of vertical automorphisms of P preserving the connection is isomorphic to $C\Phi$, the centraliser of Φ in G.

Proof. See [6] and [7].

In physical language: the existence of a non-trivial gauge configuration in the vacuum breaks the gauge group to the centraliser of the holonomy group. In order, then, to break a group G to a specified subgroup H, we must find a subgroup J with $C\mathrm{J} = \mathrm{H}$. The first question, then, is: does J exist?

2.1. Technical preliminaries

We collect here some general results on centralisers. When a result follows in some straightforward way from a definition, the proof has been deleted. Note that, while the concept of a centraliser is well defined for subsets of a group as well as for subgroups, we are only interested in the subgroup case (since J must ultimately be represented as a holonomy *group*); therefore, the notation $\mathrm{A} \subseteq \mathrm{B}$ always means that A is a subgroup of B, where B is a group. Proper subgroups are denoted $\mathrm{A} \subset \mathrm{B}$. If A and B are subgroups of G, then $\mathrm{A} \cdot \mathrm{B}$ is defined as the subset of G consisting of all products ab, where $a \in \mathrm{A}$, $b \in \mathrm{B}$. It is a subgroup of G if and only if $\mathrm{A} \cdot \mathrm{B} = \mathrm{B} \cdot \mathrm{A}$, and henceforth we shall use this notation with the understanding that this condition is satisfied. We begin with some simple results, true for all groups.

Lemma 2. Let A and B be subgroups of G, with $\mathrm{A} \subseteq \mathrm{B}$. Then $(C_{\mathrm{B}}\mathrm{A}) \cdot C\mathrm{B} \subseteq C\mathrm{A}$.

Corollary. $C\mathrm{B} \subseteq C\mathrm{A}$.

These results lead us to ask the following question. Suppose that A is properly contained in B. Then, is $C\mathrm{B}$ properly contained in $C\mathrm{A}$? The answer is 'not necessarily' and, as we shall see, this is an observation of fundamental importance. Suppose, then, that $C\mathrm{A} = C\mathrm{B}$. How are A and B related? We have the following result.

Lemma 3. Let $\mathrm{A} \subseteq \mathrm{B} \subseteq \mathrm{G}$ and suppose $C\mathrm{A} = C\mathrm{B}$. Then $\mathrm{B} = C_{\mathrm{B}}^2\mathrm{A}$, where by definition $C_{\mathrm{B}}^2\mathrm{A} = C_{\mathrm{B}}(C_{\mathrm{B}}\mathrm{A})$.

Proof. By lemma 2, we see that $C\mathrm{B} = C\mathrm{A}$ implies $C_{\mathrm{B}}\mathrm{A} \subseteq C\mathrm{B}$. But from its definition, $C_{\mathrm{B}}\mathrm{A} \subseteq \mathrm{B}$, and so $C_{\mathrm{B}}\mathrm{A} \subseteq \mathrm{B} \cap C\mathrm{B} = Z\mathrm{B}$. But by the definition of $Z\mathrm{B}$, we have $Z\mathrm{B} \subseteq C_{\mathrm{B}}\mathrm{A}$, and so $C_{\mathrm{B}}\mathrm{A} = Z\mathrm{B}$. Thus $C_{\mathrm{B}}^2\mathrm{A} = C_{\mathrm{B}}Z\mathrm{B} = \mathrm{B}$.

The significance of this result is that the statement $C_{\mathrm{B}}^2\mathrm{A} = \mathrm{B}$ is totally independent of the properties of G, and also of the way in which B is embedded in G. We shall use this result below. Lemma 3 suggests that we should examine the mappings C^2, C^3 and so on; we shall see later that they are of fundamental importance.

We conclude this section with some results on centres.

Lemma 4. Let H be a subgroup of G. Then $Z\mathrm{H} \subseteq ZC\mathrm{H}$.

This result is useful in the actual computation of centralisers: it tells us, for example, that the centraliser of SO(4) in SO(7) cannot be SO(3), since $Z(\mathrm{SO}(4)) = \mathscr{Z}_2$, while $Z(\mathrm{SO}(3)) = \mathscr{Z}_1$, the group consisting of the identity.

A much deeper result on centres, due to Wolf, can be motivated as follows. Let $\mathrm{H} \subseteq \mathrm{G}$. Then clearly $\mathrm{H} \subseteq CZ\mathrm{H}$, and, if H is connected, $\mathrm{H} \subseteq \mathrm{E}CZ\mathrm{H}$. When do we have equality?

Theorem 5. Let H be a connected subgroup of maximal rank in a compact Lie group G. Then H is the identity component of the centraliser of $Z\mathrm{H}$ in G; that is, $\mathrm{H} = \mathrm{E}CZ\mathrm{H}$.

Proof. See [8, p 276].

Here, 'maximal' rank means $\mathrm{rk(H)} = \mathrm{rk(G)}$; note that subgroups of maximal rank need not be maximal (i.e. there can exist a connected subgroup $\bar{\mathrm{H}}$ with $\mathrm{H} \subset \bar{\mathrm{H}} \subset \mathrm{G}$—for example, $\mathrm{SO}(10) \times \mathrm{SO}(6) \subset \mathrm{SO}(16) \subset \mathrm{E}_8$, all three groups being of rank 8), and that maximal subgroups need not be of maximal rank (for example, SO(7) (rank 3) is maximal in SO(8) (rank 4)).

2.2. Existence of J with CJ = H

The key to the existence problem is the study of the 'higher' centralisers, $C^2\mathrm{H} = C(C\mathrm{H})$, $C^3\mathrm{H} = C(C^2\mathrm{H})$ and so on. Our first result is trivial.

Lemma 6. Let $\mathrm{H} \subseteq \mathrm{G}$. Then $\mathrm{H} \subseteq C^2\mathrm{H}$.

At this point some terminology is useful. We shall say that $\mathrm{H} \subseteq \mathrm{G}$ is a centraliser in G if there exists J with $C\mathrm{J} = \mathrm{H}$. Theorem 1 implies that, in order to break a gauge symmetry G down to a subgroup H, it is necessary (though not sufficient) that H should be a centraliser in G. Lemma 6 now allows us to characterise centralisers.

Proposition 7. Let H be a subgroup of any group G. Then H is a centraliser in G if and only if $C^2\mathrm{H} = \mathrm{H}$.

Proof. If $C^2\mathrm{H} = \mathrm{H}$, then take $\mathrm{J} = C\mathrm{H}$. Conversely, suppose that there exists at least one J with $C\mathrm{J} = \mathrm{H}$. Then $C^2\mathrm{J} = C\mathrm{H}$ and so, by lemma 6, $\mathrm{J} \subseteq C\mathrm{H}$. Now, by the corollary to lemma 2, we have $C^2\mathrm{H} \subseteq C\mathrm{J} = \mathrm{H}$. But by lemma 6, $\mathrm{H} \subseteq C^2\mathrm{H}$ and so $\mathrm{H} = C^2\mathrm{H}$.

Thus, if $\mathrm{H} \neq C^2\mathrm{H}$ (i.e. in view of lemma 6, if it is a proper subgroup), then there exists no J with $C\mathrm{J} = \mathrm{H}$; an extremely simple yet powerful result. In simple cases, the condition $C^2\mathrm{H} = \mathrm{H}$ can be checked by means of a direct computation. Often, however, we can use proposition 9 derived below.

Proposition 7 suggests that it may be useful to examine $C^3\mathrm{H}$, $C^4\mathrm{H}$ and so on. In fact, these groups exhibit an interesting 'periodicity'.

Lemma 8. Let $\mathrm{H} \subseteq \mathrm{G}$. Then

$$
\begin{aligned}
C^n\mathrm{H} &= C\mathrm{H} \quad n \text{ odd.} \\
&= C^2\mathrm{H} \quad n \text{ even.}
\end{aligned}
$$

Proof. Obviously CH and C^2H are centralisers. Hence, by proposition 7, $C^3H = C^2(CH) = CH$, while $C^4H = C^2(C^2H) = C^2H$, and so on.

Thus, there is nothing new to be obtained beyond C^2H. However, lemma 8 allows us to prove the following very useful result.

Proposition 9. Let $H \subseteq G$. Then $ZH = ZCH$ is a necessary condition for H to be a centraliser in G.

Proof. By lemma 4, we have $ZH \subseteq ZCH \subseteq ZC^2H \subseteq ZC^3H$. But by lemma 8, $C^3H = CH$, so we have $ZCH = ZC^2H$ always. Now if H is a centraliser, then, by proposition 7, $H = C^2H$, so $ZCH = ZC^2H = ZH$.

Remark. In the course of the proof we found that, whether or not H is a centraliser, we always have $ZH \subseteq ZCH = ZC^nH$ for all $n \geq 1$.

This result shows that the centres of H and CH play a key role in determining whether H is a centraliser. From a practical point of view, the result often greatly simplifies the task of showing that a given subgroup is *not* a centraliser. For example, it is immediately obvious that SO(3) cannot be a centraliser in SO(7): for the centraliser of SO(3) in SO(7) must clearly be at least as large as SO(4), which already has a larger centre than SO(3). The case of SU(m) embedded in SU(n) can be treated with similar dispatch. We shall return to this in § 4.

The obvious question at this point is that of whether $ZH = ZCH$ is *sufficient*, as well as necessary, for H to be a centraliser. Intuitively, it is rather clear that this is asking too much; however, the study of this question will greatly clarify the reasons for the fact that certain subgroups are centralisers while others are not.

Before proceeding to that study, however, let us consider the case where H is a centraliser, so that $C^2H = H$. Then obviously H is the centraliser of CH. But, in general, CH is not the *only* group J with $CJ = H$; and so it may be necessary to make a choice from a set of candidates. It is important, for various reasons, to have some understanding of the range of possibilities in a given case. We raise this question of uniqueness at this point because of the following curious circumstance. If we combine lemma 3 with proposition 7, and note that every group G satisfies $G = CZG$, then the following result emerges.

Proposition 10. Let $A \subseteq B \subseteq G$ and suppose $CA = CB$. Then $A = B$ if and only if A is a centraliser in B.

One might say, then, that the non-uniqueness problem arises from the fact that not every subgroup of a given group is necessarily a centraliser in that group. On the other hand, from lemma 8 we have $CH = C^3H$, which can be written as $C(H) = C(C^2H)$. Comparing this with proposition 7, we can say that the failure of some subgroups to be centralisers arises from the fact that $CA = CB$ does not necessarily imply $A = B$—that is, from non-uniqueness. So we have a kind of duality between the existence and uniqueness problems for 'solutions' of the equation $CJ = H$. This means that techniques for solving either problem can usually be applied to the other. We shall therefore postpone the question of whether $ZH = ZCH$ is sufficient for H to be a centraliser until we have some more information on the consequences of $CA = CB$.

Without further information on embedding (see below), it is not possible to obtain very precise results. One expects, however, that if $A \subseteq B \subseteq G$ and $CA = CB$, then A cannot be much 'smaller' than B. In the case of Lie groups, this can be made more precise as follows. Recall that, if A and B are connected, then A is said to be maximal in B if there exists no connected group D with $A \subset D \subset B$. (Note the word 'connected': in some cases the concept of 'maximality' does not make sense without this restriction.) We shall say that A is semi-maximal in B if there exists no connected subgroup D of maximal rank (in B) with $A \subset D \subset B$. Obviously, every maximal subgroup is semi-maximal, but the converse is not true. When B is semisimple, however, counterexamples are rather uncommon. (An easily analysed case is provided by embedding the algebra of the exceptional group G_2 in the algebra of SO(8), as in table 14 of [9]. Since G_2 itself is the only [10] connected group with this algebra, we have in fact an embedding of G_2 as a subgroup of SO(8). Now G_2 is not contained in any maximal rank proper subgroup of SO(8), but it is not maximal, because it is contained in the (maximal) SO(7) subgroup of SO(8).) Thus, 'semi-maximality' still implies a close relationship between A and B.

We now have the following result.

Proposition 11. Let $A \subseteq B$ be compact, connected subgroups of a Lie group G, with $CA = CB$. Then A is semi-maximal in B. If (rank B) − (rank A) $\leqslant 1$, then A is maximal in B; if rank A = rank B, then A = B.

Proof. Combining lemma 3 with the remark after proposition 9, we have $ZA \subseteq ZC_B^2A = ZB$. Now let D be a connected group with $A \subseteq D \subseteq B$. Then D is compact, and by the corollary to lemma 2 we have $CB \subseteq CD \subseteq CA$, so $CA = CD = CB$. By the above reasoning, $ZA \subseteq ZD \subseteq ZB$. Now it follows that $B = C_BZB \subseteq C_BZD$, so in fact $B = C_BZD$. Since B is connected, $B = EC_BZD$. Now theorem 5 gives B = D if D is of maximal rank in B, and so A is semi-maximal in B. If (rank B) − (rank A) $\leqslant 1$, then any D with $A \subseteq D \subseteq B$ must have either rank D = rank A or rank D = rank B. Repeating the above argument, we have either D = A or D = B, so A is maximal in B. Finally, the same reasoning shows that if rank A = rank B, then A = B.

Setting $A = U(1)$, $B = U(1)^3$ $(= U(1) \times U(1) \times U(1))$, and $G = U(1)^4$, one sees that this result cannot be improved unless further conditions are imposed; thus proposition 11 is the basic result on the consequences of the equation $CA = CB$. By the 'duality' mentioned earlier, it is also the basic result for further investigations of the conditions under which $C^2H = H$. In order to proceed, we must now discuss the question of embeddings.

2.3. Choosing the embedding

The group $U(1) \times SU(3)$ can be regarded as a subgroup of SU(6). What is its centraliser? Unfortunately, this question cannot be answered until the embedding is specified. For example, if we choose the most natural embedding,

$$(\alpha, s) \rightarrow \begin{bmatrix} \alpha I_3 & 0 \\ 0 & \alpha^{-1}s \end{bmatrix}$$

where $\alpha \in U(1)$, $s \in SU(3)$ and I_3 is the 3×3 identity matrix, then the centraliser is isomorphic to $U(1)\times SU(3)$. But if we choose the embedding

$$(\alpha, s) \to \begin{bmatrix} 1 & 0 & 0 & \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha^2 & \\ & 0 & & \alpha^{-1}s \end{bmatrix}$$

then the centraliser will be quite different. Similarly, the centraliser of SU(3) in E_8 is E_6 only if the 'most natural' embedding is chosen.

Clearly, the natural embeddings of H in G are those for which the centraliser is as large as possible. The usual way of ordering Lie groups (in the Cartan classification, for example) is by their rank. We shall therefore use $\mathrm{rk}\,C\mathrm{H}$ as an (admittedly very rough) measure of the 'size' of $C\mathrm{H}$. Now given any two groups G and H (not necessarily a subgroup of G), define a quantity $\mu_G(\mathrm{H})$ by

$$\mu_G(\mathrm{H}) \equiv \mathrm{rkG} - \mathrm{rkH} + \mathrm{rk}Z\mathrm{H}.$$

This quantity has the property that, if H should be a subgroup of G, then $\mu_G(\mathrm{H})$ is entirely independent of the embedding—it depends only on the intrinsic (invariant) properties of G and H. The importance of this particular combination of invariants derives from the following lemma.

Lemma 12. Let H be a subgroup of a Lie group G, embedded in any way. Then $\mathrm{rk}C\mathrm{H} \leqslant \mu_G(\mathrm{H})$.

Proof. Clearly $\mathrm{H}\cdot C\mathrm{H} = (C\mathrm{H})\cdot \mathrm{H}$, so $\mathrm{H}\cdot C\mathrm{H}$ is a subgroup of G. Thus $\mathrm{rk}(\mathrm{H}\cdot C\mathrm{H}) \leqslant \mathrm{rkG}$. The result now follows from an elementary computation, using $Z\mathrm{H} = \mathrm{H}\cap C\mathrm{H}$.

The quantity $\mu_G(\mathrm{H})$ thus places an *invariant* upper bound on the 'size' of $C\mathrm{H}$. In a surprising number of cases, these very elementary considerations give the best possible upper bound, in the sense that there exists an embedding with $\mathrm{rk}C\mathrm{H} = \mu_G(\mathrm{H})$; the exceptional cases cause no problems.

The idea that $C\mathrm{H}$ should be as large as possible can now be interpreted to mean that $\mathrm{rk}C\mathrm{H} = \mu_G(\mathrm{H})$. In such a case we shall say that the embedding is *satisfactory.* For example, the usual embedding of SU(3) in E_8 is satisfactory, since $\mu_G(\mathrm{H}) = 8-2+0=6$, which equals the rank of E_6. The property of being satisfactory is not, in fact, particularly restrictive: for example, both embeddings of $U(1)\times SU(3)$ in SU(6) given above are satisfactory. Without some such restriction, the range of possibilities for $C\mathrm{H}$ (given G and H) is excessive. For example, with $G = E_8$, $H = SU(2)$, we have $\mu_G(\mathrm{H}) = 7$. There is an embedding with $\mathrm{rk}C\mathrm{H} = 7$; but, at the opposite extreme, there is another with $\mathrm{rk}C\mathrm{H} = 0$.

If we confine our attention to satisfactory embeddings, it becomes possible to clarify, for example, the conditions under which $\mathrm{H} \subseteq \mathrm{G}$ is a centraliser. We saw earlier (proposition 9) that $Z\mathrm{H} = ZC\mathrm{H}$ is *necessary* for H to be a centraliser. Is it sufficient? Unhappily, the answer is 'not quite', as the following result shows.

Proposition 13. Let H be a connected subgroup, via a satisfactory embedding, of a compact Lie group G. Then if $Z\mathrm{H} = ZC\mathrm{H}$, H is the identity component of $C^2\mathrm{H}$.

Proof. From lemma 6 and the fact that H is connected, we have $\mathrm{H} \subseteq \mathrm{E}C^2\mathrm{H} \subseteq C^2\mathrm{H}$. By the corollary to lemma 2, we now find $C^3\mathrm{H} \subseteq C\mathrm{E}C^2\mathrm{H} \subseteq C\mathrm{H}$ and so by lemma 8, $C\mathrm{H} = C(\mathrm{E}C^2\mathrm{H})$.

Now as mentioned at the end of § 1, we always take Lie subgroups to be closed. It can be shown [7] that for any subgroup (indeed, subset) S of a topological group G, $C\mathrm{S}$ is always closed. Since G is compact, it follows that H, $C\mathrm{H}$, $C^2\mathrm{H}$ and $\mathrm{E}C^2\mathrm{H}$ are all compact. Now since H and $\mathrm{E}C^2\mathrm{H}$ have the same centraliser, it follows from proposition 11 that H is semi-maximal in $\mathrm{E}C^2\mathrm{H}$. But we can do better than this. We have $\mathrm{rk}C\mathrm{H} = \mu_{\mathrm{G}}(\mathrm{H})$, so $\mathrm{rkH} = \mathrm{rkG} - \mathrm{rk}C\mathrm{H} + \mathrm{rk}Z\mathrm{H} = \mathrm{rkG} - \mathrm{rk}C\mathrm{H} + \mathrm{rk}ZC\mathrm{H}$ since $Z\mathrm{H} = ZC\mathrm{H}$ by hypothesis. Thus $\mathrm{rkH} = \mu_{\mathrm{G}}(C\mathrm{H})$, and so, by lemma 12, $\mathrm{rkH} \geqslant \mathrm{rk}C^2\mathrm{H}$. But H is a subgroup of $C^2\mathrm{H}$, so $\mathrm{rkH} \leqslant \mathrm{rk}C^2\mathrm{H}$, whence $\mathrm{rkH} = \mathrm{rk}C^2\mathrm{H} = \mathrm{rkE}C^2\mathrm{H}$. Then proposition 11 gives $\mathrm{H} = \mathrm{E}C^2\mathrm{H}$, as required.

Remark. Clearly, the condition $Z\mathrm{H} = ZC\mathrm{H}$ can be weakened to $\mathrm{rk}Z\mathrm{H} = \mathrm{rk}ZC\mathrm{H}$. This will be useful below.

Thus if H is connected and $C^2\mathrm{H}$ is not, then H fails to be a centraliser even though $Z\mathrm{H} = ZC\mathrm{H}$ may be satisfied. For example, the centraliser of SO(4) in SO(7), with the obvious embedding, is isomorphic to $\mathscr{Z}_2 \times \mathrm{SO}(3)$. Since $Z(\mathrm{SO}(4)) = \mathscr{Z}_2$ and SO(3) has trivial centre, we have $Z(\mathrm{SO}(4)) = ZC(\mathrm{SO}(4))$ but in fact $C^2(\mathrm{SO}(4)) = \mathrm{O}(4)$. The identity component is indeed SO(4), in agreement with proposition 13 (the embedding is satisfactory), but the fact remains that SO(4) is not a centraliser in SO(7); indeed, $\mathrm{SO}(m)$ is never a centraliser in $\mathrm{SO}(n)$, for all n and $2 < m < n$. That is, the Hosotani mechanism cannot break $\mathrm{SO}(n)$ to $\mathrm{SO}(m)$ (unless $m = 2$).

Proposition 13 suggests that, in many cases, it will be difficult to use the Hosotani mechanism to break connected groups down to connected subgroups; it may be easier to obtain *disconnected* gauge groups. For example, if $s \in \mathrm{O}(4)$, let $d = \det(s)$ and embed O(4) in SO(7) by mapping s to $[\begin{smallmatrix} s & 0 \\ 0 & dI_3 \end{smallmatrix}]$. Then $C^2(\mathrm{O}(4)) = \mathrm{O}(4)$, so O(4) is a centraliser in SO(7). As we shall see later, O(4) also satisfies the other conditions for the Hosotani mechanism to apply. Thus we reach the remarkable conclusion that, while SO(7) cannot be broken to SO(4), it *can* be broken to O(4). This suggests that the possibility of disconnected gauge groups [11, 12] should be investigated further.

2.4. *The question of uniqueness*

Referring again to theorem 1, we see that in order to break G to H, it is not enough merely to find J with $C\mathrm{J} = \mathrm{H}$: we must also find a principal bundle with a connection having J as holonomy group. Anticipating the results of § 3, it can be shown that such a bundle nearly always exists if J is connected; so among the various 'solutions' of $C\mathrm{J} = \mathrm{H}$, the connected J (if any) may be of particular interest. In other cases, the zero-dimensional J (if any) are of interest. In short, we need some control over the range of possible J. The basic result is as follows.

Lemma 14. Let J be any subgroup of Lie group G such that $C\mathrm{J} = \mathrm{H}$. Then J is a subgroup of $C\mathrm{H}$. If J is connected, then $C(\mathrm{E}C\mathrm{H}) = \mathrm{H}$.

Proof. If $C\mathrm{J} = \mathrm{H}$, then $C^2\mathrm{J} = C\mathrm{H}$ so $\mathrm{J} \subseteq C\mathrm{H}$ by lemma 6. If J is connected then $\mathrm{J} \subseteq \mathrm{E}C\mathrm{H} \subseteq C\mathrm{H}$ and so by the corollary to lemma 2, $C^2\mathrm{H} \subseteq C\mathrm{E}C\mathrm{H} \subseteq C\mathrm{J} = \mathrm{H}$. But H is a centraliser so, by proposition 7, $C\mathrm{E}C\mathrm{H} = \mathrm{H}$.

The first part of this lemma means that, in order to find all possible J, we need only examine the subgroups of CH. Now of course CH itself is a possible J; often, however, CH is disconnected, and it is frequently not possible to represent a disconnected group as the holonomy group of a connection on a bundle over a given base manifold M. If that should be the case, we can look for disconnected subgroups of CH which *can* be thus represented, or we can examine connected subgroups (for which such a representation is always possible)—taking care, of course, to verify that we still have $C\mathrm{J} = \mathrm{H}$. In this subsection we deal with the connected case, the more difficult disconnected case being postponed to § 3.

If CH is disconnected, then its largest connected subgroup is, of course, ECH. If the centraliser (in G) of ECH is H, then we have succeeded in finding a connected J with $C\mathrm{J} = \mathrm{H}$. If not, however, then the second part of lemma 14 tells us that there does not exist *any* connected J with $C\mathrm{J} = \mathrm{H}$, and so we have no choice but to consider the disconnected subgroups. The only remaining question in the connected case, therefore, is this: if the centraliser of ECH is H, is there any other connected J with $C\mathrm{J} = \mathrm{H}$? As usual, the answer depends on the embedding; in the semisimple case, the embedding is the only 'variable'.

Proposition 15. Let H be a semisimple subgroup of a compact Lie group G, and let J be a connected subgroup of G with $C\mathrm{J} = \mathrm{H}$. Then if J is satisfactorily embedded either in G or in CH, we have $\mathrm{J} = \mathrm{E}(C\mathrm{H})$.

Proof. By lemma 4, $Z\mathrm{J} \subseteq Z\mathrm{H}$ and so $0 \leqslant \mathrm{rk}Z\mathrm{J} \leqslant \mathrm{rk}Z\mathrm{H} = 0$ since H is semisimple. Thus $\mathrm{rk}Z\mathrm{J} = \mathrm{rk}ZC\mathrm{J}$ and so by the remark after proposition 13, $\mathrm{J} = \mathrm{E}C^2\mathrm{J} = \mathrm{E}C\mathrm{H}$, provided that J is satisfactorily embedded in G. If instead we know that J is satisfactorily embedded in CH, then denoting CH by K we have $\mathrm{J} \subseteq \mathrm{K}$ (lemma 14) and $C\mathrm{J} = C\mathrm{K} = \mathrm{H}$, so as above $\mathrm{rk}Z\mathrm{K} = 0$, and also $\mathrm{K} = C^2_{\mathrm{K}}\mathrm{J}$ (lemma 3). Then, by lemma 8, $C_{\mathrm{K}}\mathrm{J} = C^3_{\mathrm{K}}\mathrm{J} = C_{\mathrm{K}}\mathrm{K} = Z\mathrm{K}$, so $\mathrm{rk}ZC_{\mathrm{K}}\mathrm{J} = \mathrm{rk}Z\mathrm{K} = 0 = \mathrm{rk}Z\mathrm{J}$. Then since K is compact, we have, by the remark after proposition 13, $\mathrm{J} = \mathrm{E}C^2_{\mathrm{K}}\mathrm{J} = \mathrm{EK} = \mathrm{E}C\mathrm{H}$.

Within the realm of satisfactory embeddings, then, the identity component of CH is the unique connected J with $C\mathrm{J} = \mathrm{H}$ (if there is any such J), provided that H is semisimple. If H is not semisimple, then the problem quickly reduces to the study of its maximal semisimple subgroup; see the examples in § 4.

The disconnected case is best considered in conjunction with the problem of representing J as a holonomy group, to which we now turn.

3. Representing J as a holonomy group

The necessary and sufficient condition for a gauge group G to be 'breakable' to a subgroup H is that there should exist a subgroup J which satisfies $C\mathrm{J} = \mathrm{H}$, and which can be regarded as the holonomy group of a connection on a principal bundle over M. If such a bundle (with connection) exists, we shall say that J can be represented as a holonomy group over M. The basic result is the following slight modification of the Hano-Ozeki-Nomizu theorem.

Theorem 16. Let M be a connected paracompact manifold with $\dim(M) > 1$, and let (P, M, G) be a principal bundle over M, where G is any Lie group. Then there exists a connection on P with holonomy group isomorphic to G if and only if P is connected.

Proof. If P is connected, then it can be shown [13] that there exists a connection on P with all holonomy bundles isomorphic to P, and so with holonomy group isomorphic to G. On the other hand, if M is connected and paracompact, then the holonomy bundles of any connection are subbundles of P with structural groups isomorphic to the holonomy group; thus, if the latter is isomorphic to G, each holonomy bundle coincides with P. But by their very definition, holonomy bundles are obviously (pathwise) connected.

In order for P to be connected, it is of course by no means necessary for G to be so; but in the connected case we do have an immediate result.

Corollary. Let M be a connected paracompact manifold with $\dim(M) > 1$, and let J be a connected Lie group. Then J can be represented as a holonomy group over M.

For we take $P = M \times \mathrm{J}$. Since, in practice, M is always connected, has $\dim(M) > 1$, and is either Riemannian or pseudo-Riemannian (hence [14] paracompact), this corollary completely solves the holonomy representation problem when J is connected; in effect, every connected J can be thus represented.

Consider, for example, the case of symmetry breaking from $\mathrm{G} = \mathrm{SO}(7)$ to $\mathrm{H} = \mathrm{O}(4)$. Here (see § 4 for further details) $C\mathrm{H} = \mathscr{Z}_2 \times \mathrm{SO}(3)$. This is a case where $C\mathrm{H}$ is disconnected, so, as in § 2, we examine $\mathrm{E}C\mathrm{H} = \mathrm{SO}(3)$. Its centraliser in SO(7) is indeed precisely O(4), so we have found a connected J with $C\mathrm{J} = \mathrm{H}$. (By proposition 15, SO(3) is the only such J, unless we look at unsatisfactory embeddings.) Thus, O(4) is the centraliser of a group which can be represented as a holonomy group, and so, as claimed earlier, SO(7) can indeed be broken to O(4) in this way.

In practice, it may not be desirable or possible to use a connected J, even if one insists that H be connected. The situation as regards disconnected J, however, is just the reverse of the conclusion in the connected case: J cannot be represented as a holonomy group unless it satisfies very restrictive conditions related to the topology of M. In order to explain these, we need a few more facts about disconnected Lie groups.

Let G be any group, with a normal subgroup N. Then G is said to be an *extension* of N by G/N. It is said to be a *split* extension if there exists a group homomorphism σ: $\mathrm{G/N} \to \mathrm{G}$ with $\pi \circ \sigma =$ identity map on G/N, where π: $\mathrm{G} \to \mathrm{G/N}$ is the projection. Such a homomorphism is clearly a monomorphism, so in this case G/N can be regarded as a subgroup of G; in fact, since for every $g \in \mathrm{G}$ there exists $n \in \mathrm{N}$ with $g = \sigma(\pi g)n$, we have $\mathrm{G} = (\mathrm{G/N}) \cdot \mathrm{N}$ (which is *not* related in any simple way to $(\mathrm{G/N}) \times \mathrm{N}$ unless $\mathrm{G/N} \subseteq C\mathrm{N}$).

Now if G is a disconnected Lie group, it can be shown [15] that the identity component EG is a closed normal subgroup of G, so that G is an extension of EG by G/EG. It may or may not be a split extension. The disconnected groups commonly encountered in physical applications are, in fact, split extensions of their identity components: for example, $\mathrm{O}(n)$ is a split extension of $\mathrm{SO}(n)$ by $\mathscr{Z}_2$. (Map 1 to the identity matrix, and -1 to the matrix $\mathrm{diag}(-1, 1, 1, \ldots)$. Note that this latter matrix does not commute with all elements of $\mathrm{SO}(n)$, and we cannot always write $\mathrm{O}(n) = \mathscr{Z}_2 \times \mathrm{SO}(n)$.) A simple example in which the extension does not split is provided by the one-dimensional subgroup of SU(2) consisting of all matrices of the form $\left[\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right]$ together with those of the form $\left[\begin{smallmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{smallmatrix}\right]$. Here again $\mathrm{G/EG} = \mathscr{Z}_2$ (the Weyl group of the SU(2) algebra, in fact) but no homomorphism can map -1 to any element of the form $\left[\begin{smallmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{smallmatrix}\right]$, since no such matrix has square equal to $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

Returning now to the problem of representing J as a holonomy group, we find that a complete treatment is possible when J is a split extension.

Proposition 17. Let M be a connected paracompact manifold with $\dim(M) > 1$, and let J be a Lie group. Suppose that J is a split extension of its identity component. Then J can be represented as a holonomy group over M if and only if the fundamental group of M, $\pi_1(M)$, admits a normal subgroup ρ such that

$$\pi_1(M)/\rho = \mathrm{J}/\mathrm{EJ}.$$

Proof. Let J be the holonomy group of a connection on a principal bundle over M. Then EJ is generated by parallel transport around contractible loops in M. Parallel transport around arbitrary loops therefore gives a homomorphism from $\pi_1(M)$ onto J/EJ. (Care must be taken to modify the loop so that it is smooth; see [13, p 75].) The stated isomorphism now follows from the homomorphism theorem [16] (that is, ρ is the kernel of the above homomorphism.) On the other hand, suppose that ρ exists. According to theorem 16, the only way to proceed is to construct a connected principal bundle over M with J as structural group. This may be done as follows. Let $\tilde{M}$ be the universal covering space of M. It may be regarded as a principal bundle $(\tilde{M}, M, \pi_1(M))$ with $\pi_1(M)$ as structural group. Then, since ρ is normal, it is not hard to show that $(\tilde{M}/\rho, M, \pi_1(M)/\rho)$ is also a principal bundle over M. (It is another covering space.) Denote $\pi_1(M)/\rho$ by Δ. Then since J is a split extension, Δ is a subgroup of J, and so the structural group can be extended to J in the usual [17] way. The bundle space $P = [(\tilde{M}/\rho) \times \mathrm{J}]/\Delta$ is connected, and so the result follows from theorem 16.

Note that, if J is either connected or discrete, then it is (trivially) a split extension of its identity component, so these cases are subsumed under proposition 17.

This result basically means that, given M and J, it is rather unlikely that J can be represented as a holonomy group if it is disconnected. A familiar example of a case where a disconnected group cannot be represented as a holonomy group is as follows. Let M be a Riemannian manifold and take $\mathrm{J} = \mathrm{O}(n)$, $n = \dim(M)$. Suppose that $\pi_1(M)$ has no subgroup of index 2 (i.e. no subgroup with precisely two distinct cosets). Then since $\mathrm{O}(n)/\mathrm{SO}(n) = \mathscr{Z}_2$, we find that there exists no bundle over M with a connection having holonomy group $\mathrm{O}(n)$. In particular, the Riemannian connection on the bundle of orthonormal frames has holonomy group no larger than $\mathrm{SO}(n)$, so this bundle is reducible to an $\mathrm{SO}(n)$ bundle. Hence [18] M is orientable. Thus, for example, any Riemannian manifold with fundamental group $\mathscr{Z}_3$ is necessarily orientable, simply because of the impossibility of representing the disconnected group $\mathrm{O}(n)$ as a holonomy group over such a manifold. (See [19] for the usual proof of this fact; note that since any group can be expressed as the disjoint union of its (right or left) cosets, any subgroup of index 2 is automatically normal.)

It is important to note that, even if H can be expressed as CJ for some connected J, it may be desirable in some circumstances to express it as the centraliser of a zero-dimensional group. By lemma 14, it suffices to examine discrete subgroups of ECH; by proposition 17, such a group can be represented as a holonomy group over a connected paracompact M with $\dim(M) > 1$ if and only if it is isomorphic to $\pi_1(M)/\rho$ for some normal ρ. (Despite this last fact, these groups have the useful property of leading to gauge fields with vanishing field strength.) To take a simple example, embed $\mathrm{SO}(2) \times \mathrm{SO}(3)$, in the obvious way, in $\mathrm{SO}(5)$. This subgroup is the centraliser of $\mathrm{SO}(2)$, which is connected; but it is also equal to $C(\mathscr{Z}_n)$, where $\mathscr{Z}_n$ is the zero-dimensional subgroup of SO(2) corresponding to the nth roots of unity in U(1). Provided that $\pi_1(M)$ admits a normal subgroup ρ with $\pi_1(M)/\rho$ isomorphic to $\mathscr{Z}_n$ for some $n > 2$, we can break SO(5) to SO(2) × SO(3) by means of a connection with vanishing curvature (gauge field strength). Further examples may be found in § 4.

One final case remains to be considered: the case where J is not a split extension of its identity component EJ. Such groups have not yet arisen in actual applications, and so we shall not go into the details. From the proof of proposition 17, it is clear that it is still necessary to have $\pi_1(M)/\rho = \mathrm{J}/\mathrm{EJ}$ if J is to be represented as a holonomy group; the question is whether this is still sufficient (with the usual restrictions on M). To put the problem in perspective, let (P, M, J) be a principal bundle over M, where P is connected. (By theorem 16, this is the only case of interest.) Then P/EJ is a connected Δ bundle ($\Delta = \mathrm{J}/\mathrm{EJ} = \pi_1(M)/\rho$) over M. Thus it is a covering space of M, and so, by the uniqueness theorem [13] for covering spaces, we have $P/\mathrm{EJ} = \tilde{M}/\rho$, where $\tilde{M}$ is the universal cover of M. We conclude that any connected J bundle over M can be constructed as an EJ bundle over $\tilde{M}/\rho$. The point is that EJ is connected, and the problem of constructing such bundles (and of classifying them) is one which involves the details of the topology of M (homology groups, etc). For completely general disconnected J, then, one must expect that it will be necessary to impose further conditions on M in order to represent J as a holonomy group over M. For example, suppose that J is not a split extension of EJ but that it has the form $\mathrm{J} = \mathrm{F} \cdot \mathrm{EJ}$, where F is a finite group. (This is not a contradiction—we suppose that $\mathrm{F} \cap \mathrm{EJ}$ is non-trivial.) Assume that F (which may not be unique) can be chosen so that it is isomorphic to $\pi_1(M)$. Then J can be represented as a holonomy group over M: for $(\tilde{M}, M, \mathrm{F})$ is a connected bundle over M, and F (unlike Δ) is a subgroup of J, so (as in the proof of proposition 17) we can extend to a connected J bundle over M and use theorem 16. (For example, take the non-split extension of U(1) consisting of SU(2) matrices $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ and $\begin{bmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{bmatrix}$. Let $\mathscr{Z}_4$ be the subgroup of SU(2) generated by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\mathrm{J} = \mathscr{Z}_4 \cdot \mathrm{U}(1)$ and so this group can be represented as a holonomy group over any M with $\pi_1(M)$ isomorphic to $\mathscr{Z}_4$.) Now note that the equation $\mathrm{J} = \mathrm{F} \cdot \mathrm{EJ}$ is consistent with the usual equation $\mathrm{J}/\mathrm{EJ} = \pi_1(M)/\rho$ (for some normal ρ); for if we factor both sides of $\mathrm{J} = \mathrm{F} \cdot \mathrm{EJ}$ by EJ and use one of the isomorphism theorems [16], we have $\mathrm{J}/\mathrm{EJ} = \mathrm{F}/(\mathrm{F} \cap \mathrm{EJ})$ which indeed has the form $\pi_1(M)/\rho$ since we are assuming that F is isomorphic to $\pi_1(M)$. But since J is not a split extension of EJ, it follows that F is not a split extension of $\mathrm{F} \cap \mathrm{EJ}$. Thus we are imposing a stronger condition on the topology of M than previously: instead of merely requiring $\pi_1(M)$ to have a normal subgroup ρ, we are also requiring that $\pi_1(M)$ should not be a split extension of ρ. The problem of determining, in general, the precise conditions under which a non-split J can be represented as a holonomy group over M will not be pursued here.

4. Examples

4.1. General remarks

In this section we give a few examples of actual computations of centralisers. For connected groups, it is a relatively straightforward matter to compute the Lie algebra of the centraliser, but it is less easy to describe the global structure; a surprising variety of behaviour is possible. For disconnected groups, even the algebra of $C\mathrm{H}$ cannot be predicted, given the algebra of H. The examples have been chosen either to illustrate these points, or because of their role in applications.

It is to be emphasised, however, that our interest in the precise global structure of CH is motivated as much by physical considerations as by a desire for exactitude. Although this structure is (apparently) irrelevant to many applications, there are other situations in which it is of decisive importance. For example, it has been pointed out (see [20] for a particularly clear discussion) that the matter Lagrangian of the Weinberg-Salam theory is such that the gauge group is U(2), *not* the locally isomorphic SU(2) × U(1). Furthermore, this kind of global distinction between locally isomorphic groups can have physically important consequences: for example, it has a bearing on the monopole charges permitted by a given theory. Similarly the 'standard model' gauge group is not SU(3) × SU(2) × U(1). This is indeed fortunate: for if the group were in truth SU(3) × SU(2) × U(1), then grand unification, whether in SU(5), SO(10) or E_6 *would be a mathematical impossibility*. For (at least with the usual algebra embeddings) SU(3) × SU(2) × U(1) is a subgroup of none of these groups—the subgroup in question is locally, but not globally, isomorphic to this group. These comments have a particular relevance in the present context. For just as CH cannot be computed unless the global structure is investigated, this structure is completely and uniquely fixed in each case; hence, the result may not be entirely under our control. An example is discussed below.

4.2. SO(m) in SO(n)

We embed SO(m) in the top left-hand corner of the SO(n) matrices, $m < n$, as shown:

$$\begin{bmatrix} s & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

The structure of the centraliser depends on whether m is even or odd. If $m = 2$, then the centraliser is clearly SO(2) × SO($n-2$). If m is an even number >2, then the centraliser is the set of all matrices of the form

$$\begin{bmatrix} \pm I_m & 0 \\ 0 & s \end{bmatrix}$$

where $s \in$ SO($n-m$). This group is isomorphic to $\mathscr{Z}_2 \times$ SO($n-m$). If m is odd, the situation is different because now $-I_m$ has determinant -1, which allows us to 'compensate' for matrices with determinant -1 in the centraliser. Thus if $s \in$ O($n-m$) and $d = \det s$, the centraliser consists of matrices

$$\begin{bmatrix} dI_m & 0 \\ 0 & s \end{bmatrix}$$

and is isomorphic to O($n-m$). As noted earlier, O($n-m$) is not necessarily isomorphic to $\mathscr{Z}_2 \times$ SO($n-m$): there is an isomorphism when $n-m$ is odd, but not when it is even.

The centre of SO(m) is SO(2) when $m = 2$, $\mathscr{Z}_2$ when m is even >2, and $\mathscr{Z}_1$ when m is odd [21]. Clearly, the centre of each of the above centralisers contains the centre of the relevant SO(m); note that the centre of O(n) is $\mathscr{Z}_2$ for all n (including $n = 2$). This fact is in agreement with lemma 4.

We now wish to ask: for which m and n, if any, is SO(m) a centraliser in SO(n)? The following table may be helpful; data for O(m) have also been given. (Here O(m) is embedded as

$$\begin{bmatrix} s & 0 \\ 0 & dI_{n-m} \end{bmatrix} \qquad \begin{bmatrix} s & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & I_{n-m-1} \end{bmatrix}$$

depending on whether $n-m$ is odd or even respectively.)

		Centralisers in SO(n)					
m	n	$Z(\mathrm{SO}(m))$	$ZC(\mathrm{SO}(m))$	$C^2(\mathrm{SO}(m))$	$Z(\mathrm{O}(m))$	$ZC(\mathrm{O}(m))$	$C^2(\mathrm{O}(m))$
2	even	SO(2)	$\mathscr{Z}_2\times\mathrm{SO}(2)$		$\mathscr{Z}_2$	$\mathscr{Z}_2\times\mathscr{Z}_2$	
2	odd	SO(2)	SO(2)	SO(2)	$\mathscr{Z}_2$	$\mathscr{Z}_2$	O(2)
even > 2	even	$\mathscr{Z}_2$	$\mathscr{Z}_2\times\mathscr{Z}_2$		$\mathscr{Z}_2$	$\mathscr{Z}_2\times\mathscr{Z}_2$	
even > 2	odd	$\mathscr{Z}_2$	$\mathscr{Z}_2$	O(m)	$\mathscr{Z}_2$	$\mathscr{Z}_2$	O(m)
odd	even	$\mathscr{Z}_1$	$\mathscr{Z}_2$		$\mathscr{Z}_2$	$\mathscr{Z}_2$	O(m)
odd	odd	$\mathscr{Z}_1$	$\mathscr{Z}_2$		$\mathscr{Z}_2$	$\mathscr{Z}_2\times\mathscr{Z}_2$	

Here we have not bothered to compute $C^2(\mathrm{SO}(m))$ or $C^2(\mathrm{O}(m))$ when the centre, $Z\mathrm{H}$, is a proper subgroup of $ZC\mathrm{H}$, for we know that $Z\mathrm{H}=ZC\mathrm{H}$ is a necessary condition for H to be a centraliser (proposition 9). In every case but one, this condition proves to be sufficient. Of the four cases with $C^2\mathrm{H}=\mathrm{H}$, only the case $\mathrm{H}=\mathrm{SO}(2)$, $n=$ odd, has a connected $C\mathrm{H}$. All of the others have $C\mathrm{H}$ disconnected, but they can be expressed as the centraliser of the identity component of $C\mathrm{H}$: we have $C(\mathrm{O}(m))=\mathscr{Z}_2\times\mathrm{SO}(n-m)$ when m is even >2 and n is odd, but $\mathrm{O}(m)=C(\mathrm{SO}(n-m))$; and $C(\mathrm{O}(m))=\mathrm{O}(n-m)$ when m is odd and n is even, but $\mathrm{O}(m)=C(\mathrm{SO}(n-m))$ once again; finally, we have O(2), which, when n is odd, has centraliser $\mathscr{Z}_2\times\mathrm{SO}(n-2)$, but $\mathrm{O}(2)=C(\mathrm{SO}(n-2))$. Note that $\mathscr{Z}_2\times\mathrm{SO}(n-2)$ provides an example (when n is odd) of a group H which is a centraliser of a disconnected group (namely O(2)) but not of any connected group; for in this case $CEC\mathrm{H}=C(\mathrm{SO}(2))=\mathrm{SO}(2)\times\mathrm{SO}(n-2)\neq\mathrm{H}$, so by lemma 14 there exists no connected group of which H is the centraliser.

The principal conclusion to be drawn from the table, however, is the fact that SO(n) cannot be broken by this mechanism to any SO(m) unless $m=2$ and n is odd. Other subgroups (such as $\mathrm{SO}(2)\times\mathrm{SO}(n-2)$) can, however, be obtained, but we shall not enter upon this here.

4.3. *SU(m) and U(m) in SU(n)*

We can embed both SU(m) and U(m) in SU(n), $n>m$; but whereas SU(m) has an obvious natural embedding, there is no such obvious choice in the case of U(m), even if we consider satisfactory embeddings only (as we shall always do henceforth). These groups are of interest both in physics and in geometry: in particular, they arise as holonomy groups of Kähler manifolds. Recently it has been pointed out [22] that the disconnected groups $\mathrm{S}_j\mathrm{U}(n)$ (defined, for any integer $j>1$, as the subgroup of U(n) consisting of matrices A with $\det A\in\mathscr{Z}_j$) can also occur as the holonomy groups of Ricci-flat Kähler manifolds, so we shall also study their centralisers.

We begin with SU(m) embedded in SU(n) as

$$\begin{bmatrix} s & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

If $n=m+1$, the centraliser consists of matrices of the form

$$\begin{bmatrix} \alpha I_m & 0 \\ 0 & \alpha^{-m} \end{bmatrix} \qquad \alpha\in\mathrm{U}(1)$$

and is obviously isomorphic to U(1). If $n > m+1$, the situation is much more subtle: the centraliser consists of matrices of the form

$$\begin{bmatrix} \alpha I_m & 0 \\ 0 & u \end{bmatrix} \qquad \alpha \in \mathrm{U}(1),\, u \in \mathrm{U}(n-m),\, \alpha^m \det u = 1.$$

While it is rather clear that this group is locally isomorphic to $\mathrm{U}(n-m)$, the existence of a global isomorphism is by no means obvious, because the equation $\alpha^m \det u = 1$ has no unique solution for α, given u. We shall now briefly indicate how to use the homomorphism theorem [16] of elementary group theory to solve this kind of problem.

The group in question consists of pairs (α, u). Any $u \in \mathrm{U}(n-m)$ can be expressed as βs, $\beta \in \mathrm{U}(1)$, $s \in \mathrm{SU}(n-m)$, and so we can write $(\alpha, u) = (\alpha, \beta I_{n-m})(1, s)$, where $\alpha^m \beta^{n-m} = 1$. Thus, the group has the form $\mathrm{D} \cdot \mathrm{SU}(n-m)$, where D is the set of pairs (α, β) with $\alpha^m \beta^{n-m} = 1$. Now the map $(\alpha, \beta) \to \alpha\beta$ is a homomorphism of D to U(1). Its kernel is the subgroup of D consisting of pairs (α, α^{-1}), where $\alpha^{2m-n} = 1$; leaving aside the case $n = 2m$, this subgroup is isomorphic to $\mathscr{Z}_{|2m-n|}$. The homomorphism covers U(1) completely provided that, for any $\gamma \in \mathrm{U}(1)$, the equations $\alpha\beta = \gamma$, $\alpha^m \beta^{n-m} = 1$ have at least one solution pair. Again, this is the case provided $n \neq 2m$. The homomorphism theorem now gives $\mathrm{D}/\mathscr{Z}_{|2m-n|} = \mathrm{U}(1)$, so that (since D is connected) D is isomorphic to U(1) itself. Thus when $n \neq 2m$, the centraliser of $\mathrm{SU}(m)$ in $\mathrm{SU}(n)$ is $\mathrm{U}(1) \cdot \mathrm{SU}(n-m) = \mathrm{U}(n-m)$ globally.

Strangely, however, the answer is different when $n = 2m$. In this case, the map $(\alpha, u) \to (\alpha, \alpha u)$ from $C(\mathrm{SU}(m))$ to $\mathrm{U}(1) \times \mathrm{SU}(m)$ is an isomorphism (since $\det \alpha u = \alpha^m \det u = 1$). Our final result, then, is that the centraliser of $\mathrm{SU}(m)$ in $\mathrm{SU}(n)$ is given by

$$\begin{aligned} C(\mathrm{SU}(m)) &= \mathrm{U}(n-m) \qquad && n \neq 2m \\ &= \mathrm{U}(1) \times \mathrm{SU}(m) && n = 2m. \end{aligned}$$

(We remind the reader that $\mathrm{U}(n) = \mathrm{U}(1) \cdot \mathrm{SU}(n) = [\mathrm{U}(1) \times \mathrm{SU}(n)]/\mathscr{Z}_n$, where $\mathscr{Z}_n = Z(\mathrm{SU}(n))$.) The break in the pattern as n passes through $2m$ is a remarkable illustration of the rich topological structure underlying this approach to symmetry breaking.

We can draw two immediate conclusions from these results. Since $Z(\mathrm{SU}(m)) = \mathscr{Z}_m$ while $ZC(\mathrm{SU}(m)) = \mathrm{U}(1)$ (if $n \neq 2m$) or $\mathrm{U}(1) \times \mathscr{Z}_m$ (if $n = 2m$), we have (directly from proposition 9) the fact that $\mathrm{SU}(m)$ is the centraliser of no subgroup of $\mathrm{SU}(n)$; that is, $\mathrm{SU}(n)$ cannot be broken to $\mathrm{SU}(m)$ by means of this mechanism, for all n and $m < n$. Secondly, it is now obvious that $\mathrm{U}(n-m)$ *is* a centraliser is $\mathrm{SU}(n)$ if $n \neq 2m$—it is the centraliser of the connected group $\mathrm{SU}(m)$ (or of U(1) if $m = 1$). Thus $\mathrm{SU}(n)$ can be broken to $\mathrm{U}(n-m)$ for such m, n. When $n = 2m$, however, we obtain not $\mathrm{U}(m)$ but rather $\mathrm{U}(1) \times \mathrm{SU}(m)$. Can $\mathrm{SU}(2m)$ be broken to $\mathrm{U}(m)$? Certainly $\mathrm{U}(m)$ can be embedded in $\mathrm{SU}(2m)$, as follows. Set $\delta = (\det u)^{-1}$ for each $u \in \mathrm{U}(m)$, and map u to

$$\begin{bmatrix} u & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & I_{m-1} \end{bmatrix} \in \mathrm{SU}(2m).$$

But the centraliser of this subgroup is the set of all matrices

$$\begin{bmatrix} \alpha I_m & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & u \end{bmatrix} \qquad \alpha \in \mathrm{U}(1),\, u \in \mathrm{U}(m-1),\, \beta = (\alpha^m \det u)^{-1}$$

a group globally isomorphic to $U(1)\times U(m-1)$. (We ignore the case $m=1$; it can be treated similarly.) Now clearly $Z(U(m))=U(1)\neq ZC(U(m))=U(1)\times U(1)$ and so $U(m)$ is not a centraliser in $SU(2m)$, at least not with this embedding. It can be shown, in fact, that $U(m)$ is never a centraliser in $SU(2m)$ except when $m=1$. Thus we have the remarkable result that the Weinberg-Salam group U(2) (for example) can be obtained from SU(5) but not from SU(4). On the other hand, $SU(2)\times U(1)$ can be obtained from SU(4) but not from SU(5). The importance of a precise description of the gauge group should be clear from these examples.

Finally, we give a very brief discussion of $S_jU(m)$, which we embed in $SU(n)$ in precisely the same way that $U(m)$ was embedded in $SU(2m)$ above. One can show that

$$\begin{aligned} C(S_jU(m)) &= U(1)\times U(n-m-1) && \text{if } n>m+1 \\ &= U(1) && \text{if } n=m+1. \end{aligned}$$

Since $Z(S_jU(m))=\mathscr{Z}_{jm}$, it is clear that $S_jU(m)$ is never a centraliser in $SU(n)$. More importantly, note that, even though $S_jU(m)$ has the same identity component as $SU(m)$, its centraliser in $SU(n)$ is different (except when $n=m+1$).

4.4. *Gauge symmetry breaking in electroweak, grand unified and superstring theories*

The above examples illustrate some of the principal techniques for implementing the Hosotani mechanism. We shall now consider some actual cases of physical symmetry breaking.

Consider first the Weinberg-Salam model. Here $G=U(2)$. This group has two distinguished U(1) subgroups, consisting, respectively, of matrices of the form

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \qquad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

The electromagnetic U(1) is a 'mixture' of these, consisting of matrices of the form

$$\begin{bmatrix} e^{ia\theta} & 0 \\ 0 & e^{ib\theta} \end{bmatrix} \qquad \theta\in\mathscr{R}$$

where a and b are certain constant real numbers. If $a\neq b$ (as is always the case in actual applications) then the centraliser is the group of matrices

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{bmatrix} \qquad \theta,\phi\in\mathscr{R}$$

isomorphic to $U(1)\times U(1)$. Obviously, the condition $ZH=ZCH$ is violated, so the electromagnetic U(1) cannot be expressed as the centraliser of any subgroup of U(2) (including discrete subgroups, since proposition 9 applies to *all* groups). Unfortunately, then, the Hosotani mechanism cannot be used in the Weinberg-Salam theory. (Note that the embedding is satisfactory: $\mu_G(H)=2-1+1=\text{rk}CH$.)

A case in which the mechanism *can* be used is the SU(5) grand unified theory. Here $G=SU(5)$ and H is a subgroup locally isomorphic to $SU(3)\times SU(2)\times U(1)$. In fact H consists of matrices of the form

$$\begin{bmatrix} u & 0 \\ 0 & t \end{bmatrix} \qquad u\in U(2),\ t\in U(3),\ (\det u)(\det t)=1.$$

Setting $u=\alpha s_1$, $t=\beta s_2$, where α, $\beta\in \mathrm{U}(1)$, $s_1\in \mathrm{SU}(2)$, $s_2\in \mathrm{SU}(3)$, we have

$$\begin{bmatrix} u & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} \alpha I_2 & 0 \\ 0 & \beta I_3 \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \qquad \alpha^2\beta^3=1$$

so H is isomorphic to $\mathrm{D}\cdot[\mathrm{SU}(2)\times \mathrm{SU}(3)]$, where D is the group of pairs (α,β) with $\alpha^2\beta^3=1$. Using the homomorphism theorem as in § 4.3, one finds $\mathrm{D}=\mathrm{U}(1)$, so $\mathrm{H}=\mathrm{U}(1)\cdot[\mathrm{SU}(2)\times \mathrm{SU}(3)]=[\mathrm{U}(1)\times \mathrm{SU}(2)\times \mathrm{SU}(3)]/\mathscr{Z}_6$, as in [20]. (Note that $\mathscr{Z}_2\times\mathscr{Z}_3=\mathscr{Z}_6$.) Here $C\mathrm{H}$ is isomorphic to U(1): it consists of matrices

$$\begin{bmatrix} \alpha I_2 & 0 \\ 0 & \beta I_3 \end{bmatrix} \qquad \alpha^2\beta^3=1.$$

(Thus the embedding is satisfactory, since $\mu_{\mathrm{G}}(\mathrm{H})=4-4+1=\mathrm{rk}C\mathrm{H}$.) Clearly $C^2\mathrm{H}=\mathrm{H}$, so H is a centraliser. (Another way of seeing this, useful in cases where explicit computations are not feasible, is to note that SU(5) has the property that $C\mathrm{J}$ is connected when J is connected. The result now follows from $Z\mathrm{H}=ZC\mathrm{H}$ and proposition 13.) In fact H can be expressed as the centraliser either of a connected group, U(1), or of a discrete group $\mathscr{Z}_n$, embedded as

$$\begin{bmatrix} \gamma^3 I_2 & 0 \\ 0 & \gamma^{-2} I_3 \end{bmatrix} \qquad \gamma\in\mathscr{Z}_n,\ n\neq 5.$$

Hence SU(5) can be broken to $[\mathrm{U}(1)\times \mathrm{SU}(2)\times \mathrm{SU}(3)]/\mathscr{Z}_6$ by a vacuum gauge field of vanishing field strength, provided that the underlying manifold satisfies $\pi_1(M)/\rho=\mathscr{Z}_n$ for some $n>1$, $n\neq 5$, and some normal subgroup ρ.

Other, perhaps more realistic, grand unified theories can be discussed in a similar manner. The principal technical complication is the fact—we stress it again—that all computations must be performed at the Lie group level, not in terms of the corresponding algebras. For example, SO(10) does *not* have a subgroup isomorphic to $\mathrm{SU}(4)\times \mathrm{SU}(2)\times \mathrm{SU}(2)$. Its universal covering group, Spin(10), does have such a subgroup (since Spin(6) is isomorphic [23] to SU(4), and Spin(4) is isomorphic to $\mathrm{SU}(2)\times \mathrm{SU}(2)$), but, if we remain with SO(10), the relevant subgroup is $[\mathrm{SU}(4)/\mathscr{Z}_2]\times[(\mathrm{SU}(2)\times \mathrm{SU}(2))/\mathscr{Z}_2]=\mathrm{SO}(6)\times \mathrm{SO}(4)$. Now for each n, $\mathrm{U}(n)$ has a natural embedding in $\mathrm{SO}(2n)$ given by [10] $A+\mathrm{i}B\to\left[\begin{smallmatrix} A & B \\ -B & A\end{smallmatrix}\right]$, so we have $\mathrm{U}(2)\times \mathrm{U}(3)\subset \mathrm{SO}(4)\times \mathrm{U}(3)\subset \mathrm{SO}(4)\times \mathrm{SO}(6)\subset \mathrm{SO}(10)$. As we know, the 'standard model' gauge group is a subgroup of $\mathrm{U}(2)\times \mathrm{U}(3)$, and so this is the correct subgroup chain for the left-right symmetric [24] symmetry breaking pattern for SO(10). If we denote the standard model group by H, we find that $C\mathrm{H}$ consists of matrices of the form ($\mathrm{G}=\mathrm{SO}(10)$)

$$\begin{bmatrix} \cos\theta I_2 & -\sin\theta I_2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \sin\theta I_2 & \cos\theta I_2 & & \\ \multicolumn{2}{c}{\multirow{2}{*}{0}} & \cos\phi I_3 & -\sin\phi I_3 \\ & & \sin\phi I_3 & \cos\phi I_3 \end{bmatrix}$$

and so is isomorphic to $\mathrm{SO}(2)\times \mathrm{SO}(2)$. (The embedding is satisfactory.) Hence $Z\mathrm{H}\neq ZC\mathrm{H}$ and so H is not a centraliser in SO(10); thus SO(10) cannot be broken to the standard group. (The same is true of other subgroup chains, such as the one through SU(5).) On the other hand, the group $\mathrm{U}(2)\times \mathrm{U}(3)$ has the same centraliser as H (and so does $\mathrm{SU}(2)\times \mathrm{SU}(3)$—this being one of many instances in which a non-semisimple subgroup has the same centraliser as its maximal semisimple subgroup). It is not difficult to show that $C^2(\mathrm{U}(2)\times \mathrm{U}(3))=\mathrm{U}(2)\times \mathrm{U}(3)$, so SO(10) can be broken to this group (which is locally isomorphic to $SU(3)\times SU(2)\times U(1)\times U(1)$) by the Hosotani mechanism. As in the SU(5) case, the centraliser $SO(2)\times SO(2)$ can be replaced by discrete subgroups of the form $\mathscr{Z}_n\times\mathscr{Z}_m$, n and $m>2$, provided that $\pi_1(M)$ satisfies the usual restriction.

Note that the Hosotani mechanism frequently leads to additional U(1) factors in the final gauge group. The problem, of course, is that we must satisfy $Z\mathrm{H}=ZC\mathrm{H}$, while $C\mathrm{H}$ becomes progressively larger as we consider larger G. (This suggests that we might consider embeddings which are not satisfactory. This is beyond our scope here.) Of course, the increasing size of $C\mathrm{H}$ does not necessarily increase the 'size' of $ZC\mathrm{H}$ in a simple way. For example, consider the E_6 grand unified theory [24]. This group has a maximal subgroup $SU(3)\cdot SU(3)\cdot SU(3)$ (isomorphic [25] to $[SU(3)\times SU(3)\times SU(3)]/\mathscr{Z}_3$). There are many ways of embedding $U(2)\times U(3)$ in this subgroup, but for the sake of concreteness we shall choose U(2) to be embedded in the first SU(3) as in § 4.3, and embed U(1) in the second SU(3) as

$$\begin{bmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha^{-2}\end{bmatrix}\qquad \alpha\in U(1).$$

We now have a subgroup $U(2)\cdot U(1)\cdot SU(3)=U(2)\cdot U(3)=U(2)\times U(3)$ in this case, the embedding being satisfactory as usual. Both the standard model group and $U(2)\times U(3)$ have centraliser $U(1)\cdot U(2)$. As in the case of SO(10), the centre of the centraliser is $U(1)\times U(1)$; again $U(2)\times U(3)$ can be expressed as the centraliser of a subgroup of E_6, while the standard group cannot. But now there is a new complication, arising from the fact that the centraliser of $U(2)\times U(3)$ is non-Abelian group $U(1)\cdot U(2)$. For if we wish to replace $U(1)\cdot U(2)$ by a discrete group, the latter will be non-Abelian and so cannot be represented as a holonomy group over a manifold with an Abelian fundamental group. The physical consequences of this have been described in [26].

Finally we consider, very briefly, the application of this method to superstring theory [1]. Here the initial gauge group is E_8, and one has a vacuum gauge configuration which—depending [22] on the geometry of the base manifold M—may have holonomy group SU(3), $S_jU(3)$ or U(3). The centraliser of SU(3) in E_8 is E_6; this fact is not at all obvious, but we shall not go into the details here. (It is easy to show that the identity component of $C(SU(3))$ is E_6, but connectedness is more difficult to establish.) In order to compute the centraliser s of $S_jU(3)$ and U(3), we need to select an embedding. The simplest procedure is to embed $S_jU(3)$ and U(3) in SO(6) as discussed above; then SO(6) is embedded, as in § 4.2, in SO(16), which in turn is a maximal subgroup of E_8 [10]. These embeddings are satisfactory. Both groups have the same centraliser in SO(6): it consists of matrices

$$\begin{bmatrix}\cos\theta I_3 & -\sin\theta I_3\\ \sin\theta I_3 & \cos\theta I_3\end{bmatrix}$$

and is isomorphic to SO(2). From § 4.2, the centraliser of SO(6) in SO(16) is $\mathscr{Z}_2\times SO(10)$. The centraliser of SO(16) in E_8 is $\mathscr{Z}_2$. Then applying lemma 2 (twice) one finds that the desired centralisers contain $SO(2)\cdot(\mathscr{Z}_2\times SO(10))\cdot\mathscr{Z}_2=SO(2)\cdot SO(10)$ since both $\mathscr{Z}_2$ factors are already contained in $SO(2)\cdot SO(10)$. In fact, both centralisers equal $SO(2)\cdot SO(10)$. In this case (the centraliser being a subgroup of $SO(6)\times SO(10)$) the actual global structure of $SO(2)\cdot SO(10)$ is $SO(2)\times SO(10)$.

With other embeddings, one can no doubt obtain other groups. It is impossible, however, to embed $S_jU(3)$, $j>1$, in E_8 so that the centraliser is E_6: this follows from $Z(S_jU(3)) = \mathscr{Z}_{3j}$, $Z(E_6) = \mathscr{Z}_3$, and lemma 4. Thus, although SU(3) and $S_jU(3)$ have the same identity component (and both can appear as holonomy groups of Ricci-flat Kähler manifolds), they lead to very different final gauge groups. Note finally that, whereas E_6 can be broken to the rank 5 group $U(2)\times U(3)$, the Hosotani mechanism applied to $SO(2)\times SO(10)$ yields the rank 6 group $SO(2)\times U(2)\times U(3)$, which is locally isomorphic to $SU(3)\times SU(2)\times U(1)\times U(1)\times U(1)$.

5. Conclusion

The results of § 4 make it quite clear that the Hosotani mechanism can supplement, but not replace, the Higgs mechanism. This method cannot be used in electroweak theory, and, in grand unified theories, it tends to produce extra U(1) factors in the final group. On the other hand, the fact that conventional gauge theory works for all compact Lie groups is no great virtue. On the contrary, this fact is strong evidence that conventional gauge theory needs to be combined with new procedures, so that gauge groups can be strongly restricted or even 'predicted'. The general idea that symmetry breaking should involve *centralisers* may therefore be a step in the right direction; in general, not many subgroups can satisfy even the necessary condition $ZH = ZCH$, and this is not always sufficient.

The Hosotani mechanism teaches another lesson: that the custom of neglecting the global structure of gauge groups is a luxury which more complete theories will not permit. If a theory is capable of predicting the *algebras* of the various groups, then we should suppose that it is capable of telling us whether those groups are simply connected, how many connected components they have, and so on. We have seen that the Hosotani mechanism can break SU(4) to $SU(2)\times U(1)$ but not to $U(2) = [SU(2)\times U(1)]/\mathscr{Z}_2$, that it can break SO(7) to O(4) but not to SO(4). No doubt these distinctions are unimportant for many applications; but the point is that the mechanism decides the issue for us. This kind of result is surely typical of symmetry breaking patterns in any more complete theory. Clearly, these alternative structures (including particularly the disconnected case) merit more attention than they have received hitherto.

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